Original Paper *Forma*, **36**, 35–40, 2021

Anomalous Period-Doubling Bifurcation of the Elliptic Fixed Point in the Area-Preserving Maps

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(Received August 31, 2021; Accepted October 27, 2021)

In 1968, Moser reported a new bifurcation through which the periodic orbits with period-3 or -4 appear. At present, this bifurcation is called the anomalous rotation bifurcation (ARB). The examples of ARB have been already known. Why the anomalous period-doubling bifurcation (APDB) of the elliptic fixed point does not happen in the area-preserving maps? In order to answer this question, we introduce the area preserving map T defined by C^n $(n \geq 1)$ mapping function and derive the conditions that APDB happens.
Key words: Anomalous Rotation/Period-Doubling Bifurcation, Area-Preserving Map

1. Introduction

In 1968, Moser reported a new bifurcation through which the periodic orbits with period-3 or -4 appear (Moser, 1968). At present, we say this bifurcation ”the anomalous rotation bifurcation” (ARB). The examples of ARB have been already known (Dullin-Meiss-Sterling, 2005) (see also Appendix A). The bifurcation through which the periodic orbit with period-2 appears from the elliptic fixed point is named the period-doubling bifurcation. The following question arises naturally. Does the anomalous period-doubling bifurcation of the elliptic fixed point happen? In oder to answer this question, we introduce the area-preserving map T.

$$T \ : \ y_{n+1} = y_n + f(x_n), \ x_{n+1} = x_n + y_{n+1}. \qquad (1)$$

The mapping function $f(x)$ is defined as follows.

$$f(x) = \begin{cases} f_l(x) = a(x - x^2) \ (x \leq 1), \\ f_r(x) = a(x - x^2) - b(x-1)^m \ (x \geq 1). \end{cases} \qquad (2)$$

Here, $a > 0$, $b \geq 0$, and $m \geq 2$. The map T is included in the Hénon family (Hénon, 1969). There are two fixed points $P = (0, 0)$ and $Q = (1, 0)$. The point $P = (0, 0)$ is a saddle fixed point. The other one $Q = (1, 0)$ is an elliptic fixed point at $0 < a < 4$ and is a saddle fixed point with reflection at $a > 4$. At $a = 4$, the period-doubling bifurcation of the fixed point Q occurs. In this paper, we study this period-doubling bifurcation of Q.

Here, using Fig. 1(a), we explain the regular period-doubling bifurcation. If the regular period-doubling bifurcation of the elliptic fixed point Q happens, two daughter periodic points z_0 and z_1 appear from Q. The relations $z_1 = Tz_0$ and $z_0 = Tz_1$ hold. Thus, the period of these points is 2. The mother point Q becomes the saddle point with reflection.

Next, we explain the anomalous period-doubling bifurcation (see Fig. 1(b)). The rotation number in the vicinity of Q is less than $1/2$. As the orbital point moves away from Q, its rotation number increases, becomes $1/2$ and decreases. As a result, the saddle points and the elliptic points appear through the saddle-node bifurcation at the region that the rotation number is 1/2. The period of these points is 2. Let a_c^{sn} be the critical value at which the saddle node bifurcation happens. We increase a from a_c^{sn}. Two saddle points move to Q. These points are absorbed into Q at $a = a_c^{\mathrm{pd}}$. On the other hand, two elliptic points recede from Q. The interval $[a_c^{\mathrm{sn}}, a_c^{\mathrm{pd}})$ is the anomalous parameter one. We remark that Q is a saddle with reflection at $a > a_c^{\mathrm{pd}}$. Our aim is to derive the condition that the anomalous period-doubling bifurcation of Q happens. Our results are summarized as Theorem 1.

Theorem 1.
(1) If the mapping function $f(x)$ satisfies the condition $m = 2$, the anomalous period-doubling bifurcation of the fixed point Q occurs at $b > 0$.
(2) If the mapping function $f(x)$ satisfies the condition $m = 3$, the regular period-doubling bifurcation of the fixed point Q occurs at $0 < b \leq 16$ and the anomalous period-doubling bifurcation of Q occurs at $b > 16$.
(3) If the mapping function $f(x)$ satisfies the condition $m \geq 4$, the regular period-doubling bifurcation of the fixed point Q occurs.
(4) If the mapping function $f(x)$ is the analytic $(b = 0)$, the regular period-doubling bifurcation of the fixed point Q occurs.

In §2, the mathematical tools and several notations used in §3 are introduced. In §3, Theorem 1 is proved. In §4, we give the conclusion and propose the problem to be solved.

doi:10.5047/forma.2021.005

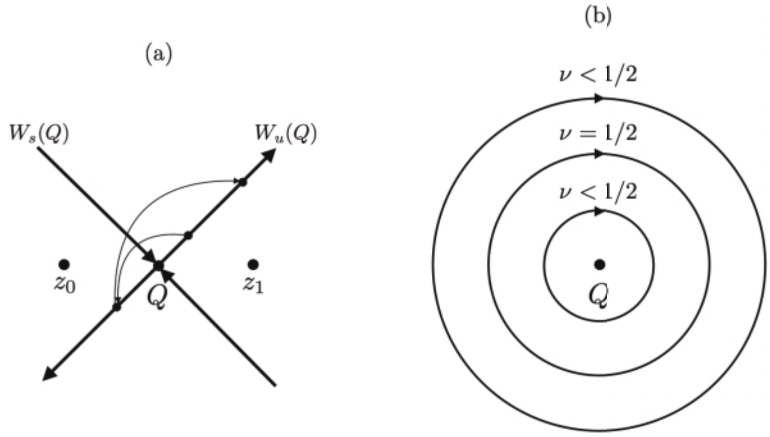

Fig. 1. (a) The situation after the regular period-doubling bifurcation ($z_1 = Tz_0,\ z_0 = Tz_1$). (b) The configuration around Q when the anomalous period-doubling bifurcation happens. Here, ν represents the rotation number.

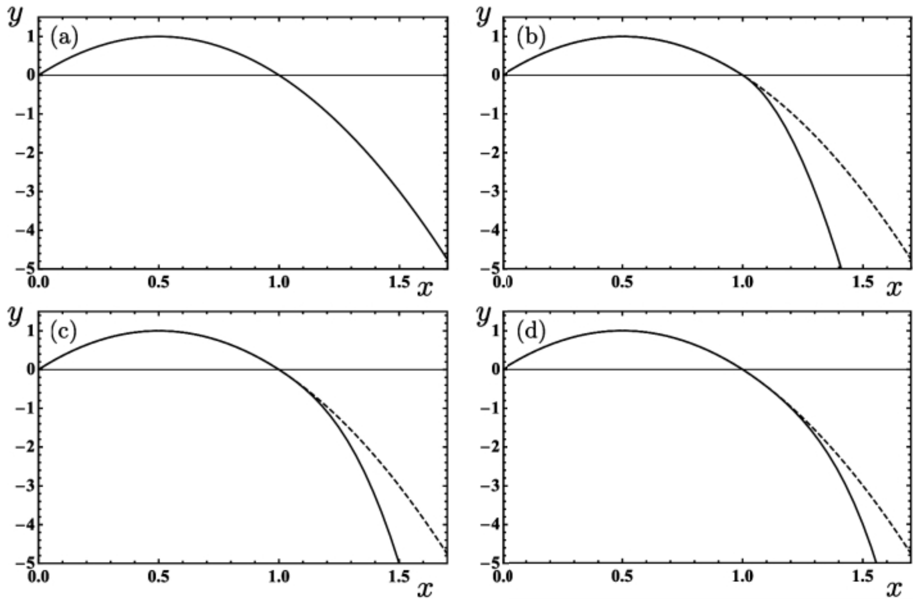

Fig. 2. (a) Analytic function ($b = 0$). (b) C^1-class function ($m = 2, b = 16$). (c) C^2-class function ($m = 3, b = 16$). (d) C^3-class function ($m = 4, b = 16$). The dotted line in the region $x > 1$ represents $y = a(x - x^2)$. $a = a_c^{\mathrm{pd}} = 4$.

2. Preliminaries

2.1 Properties of the mapping function

The properties of the mapping function $f(x)$ are summarized. The function $f(x)$ with $b = 0$ is analytic and $f(x)$ with $b > 0$ is of C^{m-1}-class.

For example, consider $f(x)$ with $m = 2$ and $b > 0$. We have $f_l'(1) = -a = f_r'(1)$ and $f''_l(1) = -2a \neq f''_r(1) = -2a - 2b$. Thus, $f(x)$ with $m = 2$ is of C^1-class.

Several mapping functions are depicted in Fig. 2. Fig. 2(a) represents the analytic function. The mapping functions with $m = 2, 3,$ and 4 are displayed in Figs. 2(b)–(d). The dotted line in the region $x \geq 1$ is $y = a(x - x^2)$. Here, we increase the value of m at the fixed value of $b > 0$ and observe that the mapping functions accumulate at the analytic one.

2.2 Critical value of the period-doubling bifurcation

The first derivative of $f(x)$ is continuous at $x = 1$. Thus, we can use it for the linear stability analysis. The linearized matrix M_Q at $Q = (1, 0)$ is obtained as

$$M_Q = \begin{pmatrix} 1 & -a \\ 1 & 1-a \end{pmatrix}. \tag{3}$$

The determinant of M_Q is 1. This means that the map is area and orientation preserving. The eigenvalues are determined by the following characteristic equation.

$$\lambda^2 - (2-a)\lambda + 1 = 0. \tag{4}$$

We have the discriminant $D = a^2 - 4a$. The fixed point Q is a stable elliptic point at $0 < a < 4$ and is a saddle point with reflection at $a > 4$. Thus, $a = 4$ is the critical value a_c^{pd} at which the period-doubling bifurcation of Q happens.

2.3 Involutions and symmetry axes

If a map is represented by the product of involutions, we say that the map has the reversibility in the sense of Birkhoff (Birkhoff, 1927). The map T is reversible. Using the involutions g and h, we represent T as $T = h \circ g$ where $h \circ h = g \circ g = \mathrm{id}$ and $\det\nabla h = \det\nabla g = -1$.

$$g\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} -y - f(x) \\ x \end{pmatrix},\ h\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} -y \\ x - y \end{pmatrix}. \quad (5)$$

The set of fixed points of involution is called the symmetry axis. Let S_g be the symmetry axis of g and S_h be the symmetry axis of h. Here, we define the portions of S_g $(y = -f(x)/2)$ and S_h $(y = 0)$.

$$S_g^+ \ : \ y = -f(x)/2 \ (x \geq 1), \quad (6)$$

$$S_g^- \ : \ y = -f(x)/2 \ (0 < x \leq 1), \quad (7)$$

$$S_h^+ \ : \ y = 0 \ (x \geq 1), \quad (8)$$

$$S_h^- \ : \ y = 0 \ (0 < x \leq 1). \quad (9)$$

In this paper, $x = 1$ is included in $S_g^{\pm}$ and in $S_h^{\pm}$.

2.4 Several maps

Differentiating the map with respect to x_n, the map of the slope $\xi_n = dy_n/dx_n$ is derived.

$$\xi_{n+1} = \frac{\xi_n + f'(x_n)}{\xi_n + f'(x_n) + 1}. \quad (10)$$

The following relation is used.

$$\frac{dy_{n+1}}{dx_n} = \frac{dy_{n+1}}{dx_{n+1}}\frac{dx_{n+1}}{dx_n} = \xi_{n+1}(\xi_n + f'(x_n) + 1). \quad (11)$$

We let $x_n = 1$ and have the one-dimensional map.

$$\xi_{n+1} = \frac{\xi_n - a}{\xi_n - a + 1}. \quad (12)$$

Differentiating Eq.(10) with respect to x_n, the map of the second derivative $\eta_n = d^2y_n/dx_n^2$ is obtained. We set $x_n = 1$.

$$\eta_{n+1} = \frac{\eta_n - 2a}{(\xi_n - a + 1)^3}. \quad (13)$$

Differentiating Eq.(10) with respect to x_n twice, the map of the third derivative $\zeta_n = d^3y_n/dx_n^3$ is obtained. We set $x_n = 1$.

$$\zeta_{n+1} = \frac{\zeta_n}{(\xi_n - a + 1)^4} - 3\frac{(\eta_n - 2a)^2}{(\xi_n - a + 1)^5}. \quad (14)$$

Here, we explain how to use Eq.(12). The slope of S_g^- at $x = 1$ is $a/2$. Using the initial condition $\xi_0 = a/2$, we calculate ξ_1 which is the slope of the image TS_g^- at $x = 1$. Let us consider the situation satisfying the condition $\xi_1 = a/2$.

$$\xi_1 = \frac{(a/2 - a)}{(a/2 - a + 1)} = \frac{a}{2}. \quad (15)$$

Solving this equation, we have the solution $a = 4$ satisfying the condition $a > 0$. This is the critical value at which the period-doubling bifurcation of Q occurs. Thus, Property 2 is derived.

Property 2. The orbital points of period-2 appearing through the period-doubling bifurcation of Q locate on S_g.

3. Proof of Theorem 1

3.1 Two criteria

The anomalous period-doubling bifurcation in the case with $m = 2$ and $b = 32$ is displayed in Fig. 3.

Fig. 3(a) represents the configuration of the symmetry axis S_g^+ (dashed line) and the image TS_g^- (solid line) at the critical situation of the saddle-node bifurcation ($a = a_c^{\mathrm{sn}} = 3.483623$). The image TS_g^- touches S_g^+ at s_1. At $a > a_c^{\mathrm{sn}}$, point s_1 changes into two points z_1 and w_1 where z_1 is the orbital point of the elliptic period-2 and w_1 is that of the saddle period-2. Here, we increase the value of a to $a = 4$. Two saddle points w_1 and w_0 (not displayed) move to Q (Fig. 3(b) and Fig. 3(c)). At $a = 4$, these points are absorbed into Q (Fig. 3(d)).

Before proceeding further, we give a remark about Arcγ $= (w_1, z_1)_{TS_g^-}$ in Fig. 3(c). Arcγ locates to the right of arc $\Gamma = (w_1, z_1)_{S_g^+}$. Suppose that $u_0 \in \Gamma$ and $u_1 = Tu_0 \in \gamma$. The rotation angle on the points of the arc connecting w_1 to u_0 is greater than π.

From the observation of Fig. 3(d) at $a = 4$, Property 3 is obtained.

Property 3.

At $a = 4$,

(i) the slope of the image TS_g^- at Q is equal to that of the symmetry axis S_g^+ at Q and

(ii) the image TS_g^- locates below S_g^+ in the right neighborhood of Q.

Using (ii) and the continuity of the image TS_g^-, the image TS_g^- intersects the symmetry axis S_g^+ outside the neighborhood of Q. The intersection point z_1 locates away from Q. This fact means that the point z_1 appears through the anomalous period-doubling bifurcation. If the image TS_g^- locates above S_g^+ in the right neighborhood of Q, there is no intersection points in the right neighborhood. Therefore, the regular period-doubling bifurcation occurs at $a = 4$.

Suppose that the image TS_g^- is represented $y = F(x)$ in the neighborhood of $x = 1$. The symmetry axis S_g^+ is represented $y = F_s(x)$ $(= -f_r(x)/2)$. Using these, we have Criterion 4.

Criterion 4.

Consider the ϵ-neighborhood of $x = 1$ at the critical situation $a = a_c^{\mathrm{pd}} = 4$.

(1) If the relation $F_s(x) < F(x)$ $(x > 1)$ holds, the regular period-doubling bifurcation occurs.

(2) If the relation $F_s(x) > F(x)$ $(x > 1)$ holds, the anomalous period-doubling bifurcation occurs.

From Criterion 4, Criterion 5 is derived. In the representation of Criterion 5, $F_s^{(k)}(1)$ means the k-th derivative of $F_s(x)$ at $x = 1$ and $F^{(k)}(1)$ means the k-th derivative of $F(x)$ at $x = 1$.

Criterion 5. At $a = a_c^{\mathrm{pd}} = 4$, the relation $F_s'(1) = F'(1)$ holds.

(1) If the relation $F_s^{(2)}(1) < F^{(2)}(1)$ holds, the regular period-doubling bifurcation occurs. If the relation $F_s^{(2)}(1) > F^{(2)}(1)$ holds, the anomalous period-doubling bifurcation occurs.

(2) Case with $F_s^{(2)}(1) = F^{(2)}(1)$. If the relation $F_s^{(3)}(1) <$

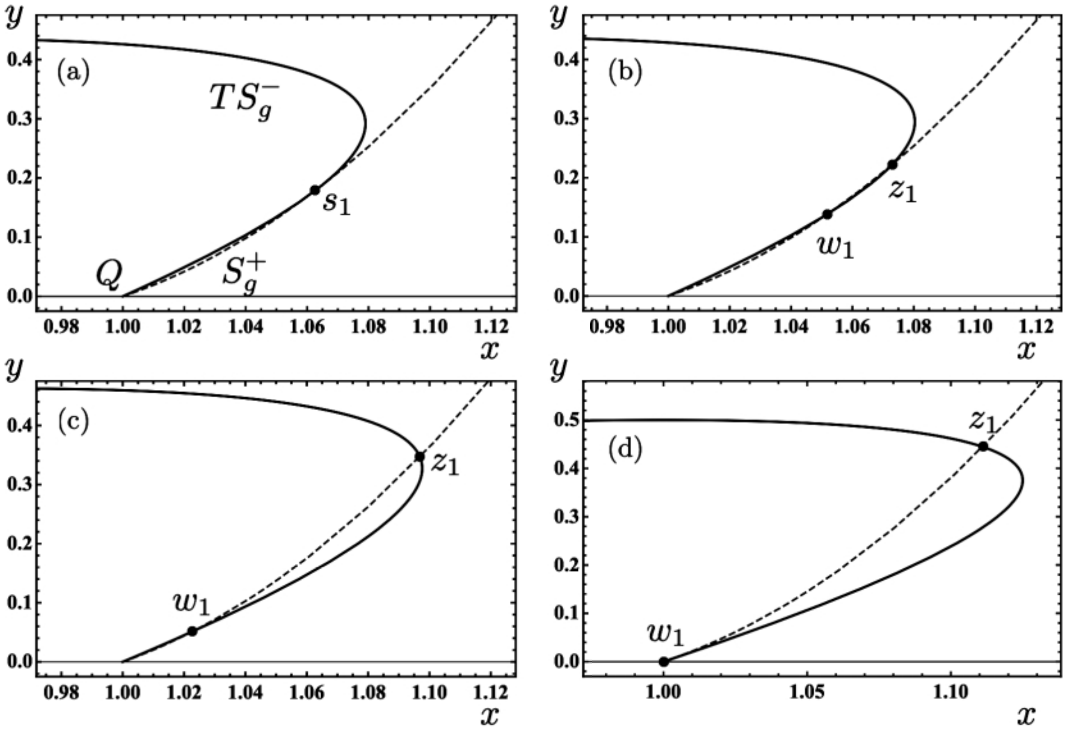

Fig. 3. Example of the anomalous period-doubling bifurcation. $m = 2$. $b = 32$. (a) $a = a_c^{sn} = 3.483623$. (b) $a = 3.5$. (c) $a = 3.7$. (d) $a = a_c^{pd} = 4$. The solid line represents the image TS_g^- and the dashed one the symmetry axis S_g^+.

$F^{(3)}(1)$ holds, the regular period-doubling bifurcation occurs. If the relation $F_s^{(3)}(1) > F^{(3)}(1)$ holds, the anomalous period-doubling bifurcation occurs.

3.2 Proof of Theorem 1

The relations of the image TS_g^- and S_g^+ around $x = 1$ are studied. First, the case with $m = 2$ is considered. In oder to determine η_1, we put $\xi_0 = a/2$, $\eta_0 = a$ and $a = 4$ in Eq.(13) where ξ_0 is the slope of S_g^- at $x = 1$ and η_0 is the second derivative of S_g^- at $x = 1$. Thus, $\eta_1 = 4$ $(= F^{(2)}(1))$ is obtained. Here, Eq.(13) is used. The second derivative of S_g^+ at $x = 1$ is $F_s^{(2)}(1) = 4 + b$. The following relation holds.

$$F_s^{(2)}(1) > F^{(2)}(1). \tag{16}$$

From Criterion 5(1), it is obtained that the anomalous period-doubling bifurcation occurs in this case. Thus, Theorem 1(1) is proved.

Next, $\eta_1 = 4$ holds in the case with $m = 3$. The second derivative of S_g^+ at $x = 1$ is also 4. Thus, we have to determine the third order derivatives. We input $\xi_0 = a/2$, $\eta_0 = a$, $\zeta_0 = 0$ and $a = 4$ in Eq.(14) and have $\zeta_1 = 48$ $(= F^{(3)}(1))$. On the other hand, the third order derivative of S_g^+ at $x = 1$ is $3b$ $(= F_s^{(3)}(1))$. From Criterion 5(2), the anomalous period-doubling bifurcation occurs if the condition $b > 16$ holds. If the condition $b < 16$ holds, the regular period-doubling bifurcation occurs

We prove that the regular period-doubling bifurcation occurs in the case with $b = 16$. In the following, we let $a = 4$. Suppose that the initial point $t_0 = (x_0, y_0)$ locates on S_g^-.

$$x_0 = t,\ y_0 = -2(t - t^2). \tag{17}$$

Here, we set $t = 1 - \epsilon$ $(\epsilon > 0)$ and obtain the position of the image $t_1 = (x_1, y_1)$.

$$x_1 = 1 + \epsilon - 2\epsilon^2,\ y_1 = 2(1 - \epsilon)\epsilon. \tag{18}$$

In order to determine the y-coordinate (y_s) of S_g^+ at $x = x_1$, we let $b = 16$.

$$y_s = 2\epsilon(1 - \epsilon - 20\epsilon^3 + 48\epsilon^4 - 32\epsilon^5). \tag{19}$$

The difference $d = y_1 - y_s$ is obtained.

$$d = y_1 - y_s = 64\epsilon^4((\epsilon - 3/4)^2 + 1/16) > 0. \tag{20}$$

From Criterion 4, it is obtained that the regular period-doubling bifurcation occurs. Thus, Theorem 1(2) is proved.

The relation $\zeta_1 = 48 = F^{(3)}(1)$ holds in the cases with $m \geq 4$. But, the third derivative of S_g^+ at $x = 1$ is $F_s^{(3)}(1) = 0$. As a result, we have the relation $F_s^{(3)}(1) < F^{(3)}(1)$. From Criterion 5(2), it is obtained that the regular period-doubling bifurcation occurs. Thus, Theorem 1(3) is proved.

Finally, the case with $b = 0$ is studied. Using the reason for the cases with $m \geq 4$, we obtain the same result that the regular period-doubling bifurcation occurs. The details are omitted. Thus, Theorem 1(4) is proved.

3.3 Examples of period-2

The orbital points of period-2 appearing through the anomalous period-doubling bifurcation are displayed in Fig. 4 ($m = 2$) and in Fig. 5 ($m = 3$). The anomalous parameter intervals are $a \in [3.852747, 4)$ ($m = 2$) and $a \in [3.924018, 4)$ ($m = 3$). The large disks represent the orbital points of elliptic period-2 and the small disks represent those of saddle ones.

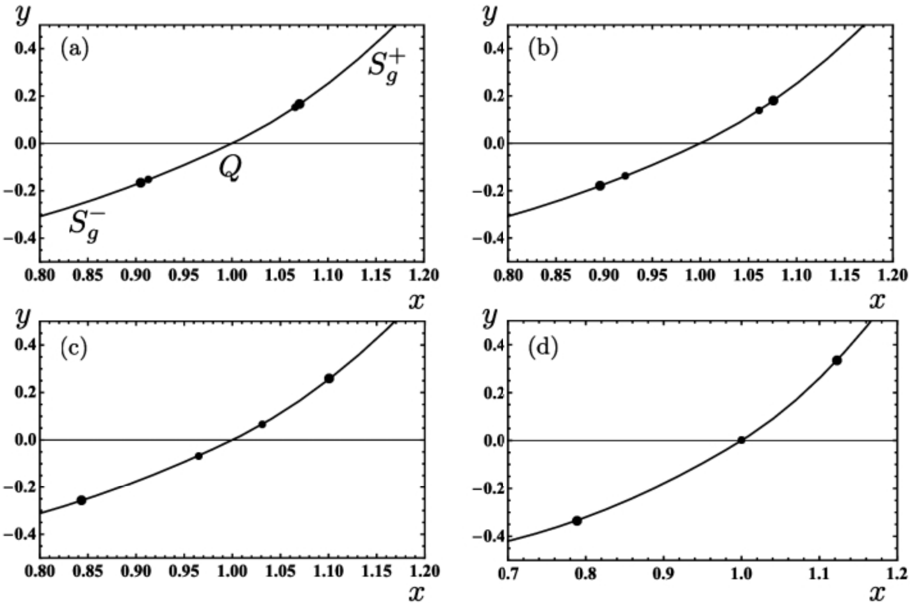

Fig. 4. $m = 2$. $b = 8$. $a_c^{\rm sn} = 3.852747$. (a) $a = 3.853$. (b) $a = 3.855$. (c) $a = 3.9$. (d) $a = 4$.

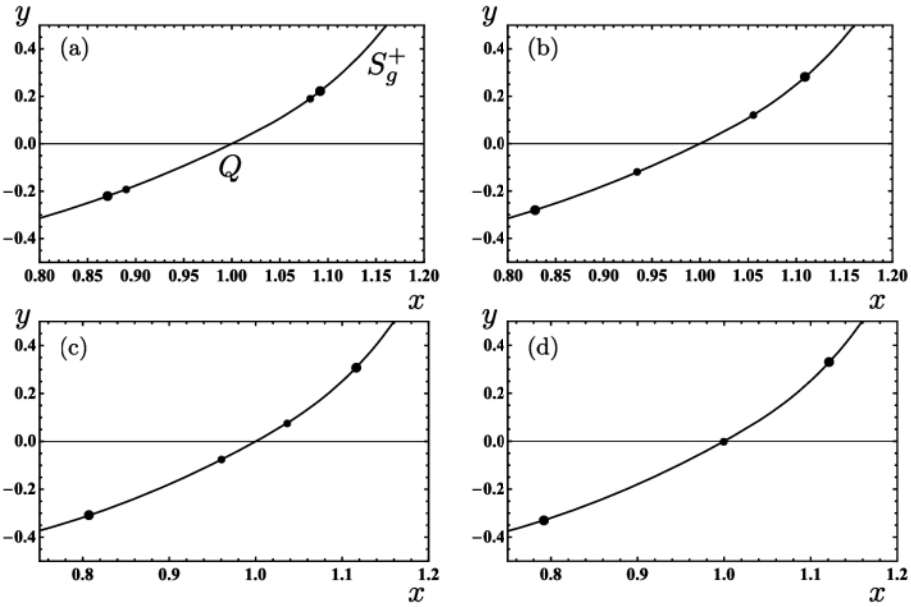

Fig. 5. $m = 3$. $b = 64$. $a_c^{\rm sn} = 3.924018$. (a) $a = 3.925$. (b) $a = 3.95$. (c) $a = 3.975$. (d) $a = 4$.

3.4 Remark

Using the Poincaré index (Guckenheimer and Holmes, 1983), we explain the fact that two saddle points absorbed into Q are not emitted from Q at $a > 4$. After the saddle-node bifurcation, the elliptic period-2 points and the saddle ones appear. Under the operation of T^2, there exist two orbital points of the elliptic period-2 and the saddle one. Thus, the Poincaré index of the elliptic points is $2 \times (+1)$. The Poincaré index of the saddle points is $2 \times (-1)$. Under the operation of T^2, Q is the elliptic point. The index is $(+1)$. Thus, the summation of the Poincaré indices is $(+1) = (+1) + (+2) + (-2)$. Before the saddle-node bifurcation, only Q exists. Thus, the summation is $(+1)$. The summation of Poincaré indices are preserved.

Next, we study the situation that two saddle points are absorbed into Q. This is the situation at $a > 4$. Thus, Q is the saddle point under the operation of T^2 and the index of Q is (-1). Under the operation of T^2, there exist two daughter elliptic points appeared from Q at $a > 4$. The Poincare index of two daughter points is $2 \times (+1)$. Suppose that two saddle points are emitted from Q at $a > 4$. The Poincare index of two saddle points is $2 \times (-1)$. Thus, we have the summation $(-1) = (-1) + (+2) + (-2)$. But, before the emission of two saddle points, the summation is $(+1)$. The contradiction is derived.

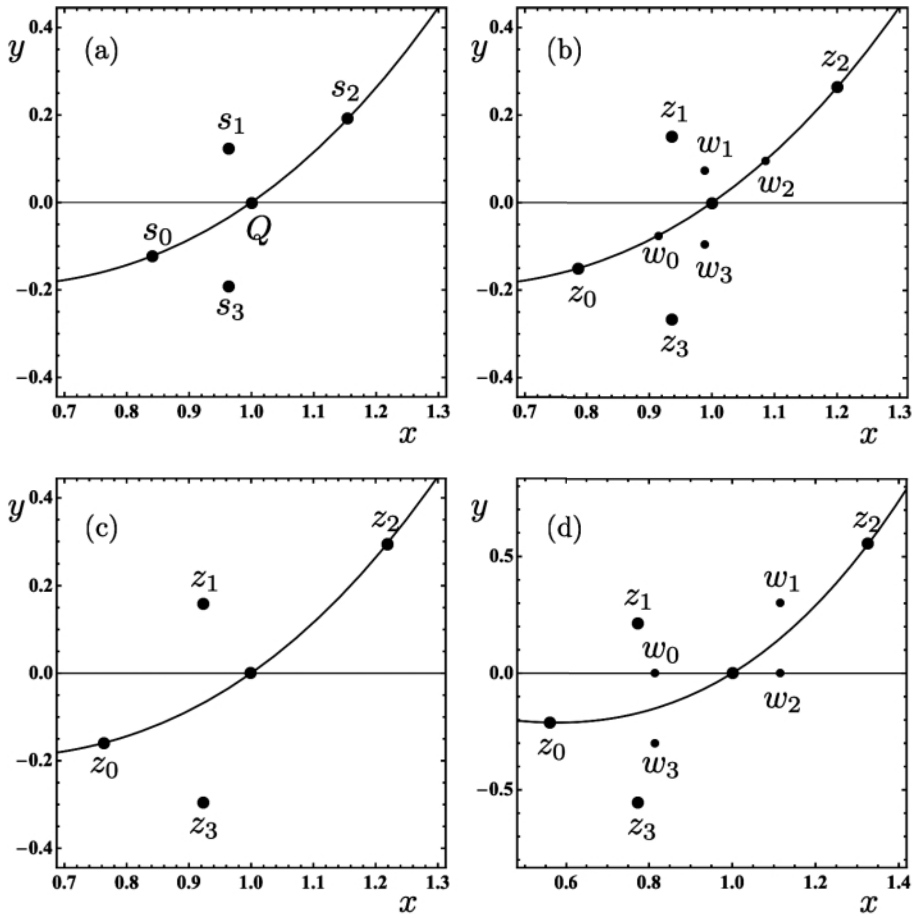

Fig. 6. (a) $a = a_c^{\mathrm{sn}} = 0.9942834$. (b) $a = 0.997$. (c) $a = a_c(1/4) = 1$. (d) $a = 1.1$.

4. Conclusion

We derive the conditions that the regular/anomalous period-doubling bifurcation of T happens. The results are summarized as Theorem 1. Theorem 1 says that the loss of smoothness of the mapping function causes the anomalous period-doubling bifurcation.

In this paper, we do not discuss the anomalous rotation bifurcation of the periodic orbit with the rotation number p/q $(0 < p/q < 1/2)$. In the cases with $m = 1, 2$ and p/q $(0 < p/q < 1/2)$, it is needed to determine the conditions at which the anomalous rotation bifurcation occurs. We leave it as a future problem.

Acknowledgments. The authors are grateful to the referee for useful comments.

Appendix A. The anomalous rotation bifurcation of period-4

We introduce the example of the anomalous rotation bifurcation of period-4. Here, the mapping function $f(x)$ is defined.

$$f(x) = a(x - x^3) \ (a > 0). \tag{A.1}$$

The anomalous parameter interval is $[a_c^{\mathrm{sn}} = 0.9942834, a_c(1/4) = 1)$. The image $T^2 S_g^-$ touches the symmetry axis S_g^+ at $a = a_c^{\mathrm{sn}}$. The tangent point s_2 locates apart from Q (see Fig. 6(a)). The two intersection points z_2 and w_2 are observed in Fig. 6(b) where z_2 is the elliptic point and w_2 is the saddle point. The saddle points w_j $(0 \le j \le 3)$ move to Q and these are absorbed into Q (Fig. 6(c)) at $a = 1$. The monotone twist property (Birkhoff, 1927) does not hold in the vicinity of Q at $a \in [a_c^{\mathrm{sn}}, a_c(1/4))$. At $a > 1$, the saddle points are emitted from Q and the point w_0 (w_2) locates on the symmetry axis S_h^- (S_h^+) (Fig. 6(d)).

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